B. Math. Hons. IInd year
Midsemestral examination
First semester 2017
Algebra III
Instructor : B. Sury
Maximum Marks 60

Q 1. (8 marks)
Give examples of the following:
(a) A commutative subring of $M_2(\mathbf{R})$;
(b) A nilpotent polynomial of positive degree in $\mathbf{Z_{12}}[X]$;
(c) A left ideal of a ring that is not a right ideal;
(d) Two nilpotent elements who sum is not nilpotent.

Q 2. (5+7 marks)
Let A be a commutative ring with unity.
(i) Prove that an ideal M is maximal if A/M is a field.
(ii) If P is a prime ideal such that A/P is a finite ring, then prove that P must be maximal.
Hint. Use the fact that if P is a prime ideal, then A/P is a domain.

Q 3. (5+5 marks)
(i) Determine, with proof, all the idempotents of the ring $R = C([0,1], \mathbf{R})$ of continuous real-valued functions on $[0,1]$.
(ii) Let A be a commutative ring with unity. If $f = a_0 + a_1X + \cdots + a_nX^n \in A[X]$ is a unit, prove that a_n is nilpotent in A.
Hint. If $g = b_0 + b_1X + \cdots + b_mX^m$ is the inverse of f, show that $a_n^{m+1-k}b_k = 0$ for $k \leq m$.

Q 4. (4+6 marks)
(i) Let R be the ring

$$\mathbf{Z}[i,j,k] = \{a + bi + cj + dk : a,b,c,d \in \mathbf{Z}\}$$

of integral quaternions. Find its group of units.
(ii) Find all square roots of -1 in the ring

$$\mathbf{H} = \{a + bi + cj + dk : a,b,c,d \in \mathbf{R}\}$$

of real quaternions.

Q 5. (12 marks)
(i) Find all units of the ring $\mathbf{Z}[\sqrt{-d}]$ where $d > 2$ is an integer.
(ii) Prove that the polynomial $X^{50} - 101101X^{13} + 110$ cannot take either of the values 33 and -33 for an integer value of X.
Hint. Apply Eisenstein's criterion to appropriate primes.

OR

(i) In a Euclidean domain, show that irreducible elements are prime.
(ii) Assuming the fact that $\mathbf{Z}[\sqrt{-2}]$ is a Euclidean domain, find all solutions in integers x, y to the equation

$$x^2 + 2 = y^3.$$

Hint for (ii). Observe that a prime p dividing y in any solution divides $(x + \sqrt{-2})(x - \sqrt{-2})$ but divides neither factor.

Q 6. (10 marks)
Consider the ring homomorphism $\phi : \mathbf{C}[X, Y] \to \mathbf{C}[Z]$ defined by $X \mapsto Z^2, Y \mapsto Z^3$. Prove that the kernel of ϕ is the principal ideal generated by $X^3 - Y^2$.

OR

Consider a ring homomorphism T from $\mathbf{R}$ to itself. Show that if T is not the zero map, T is identity on $\mathbf{Q}$ and that $T(x) \geq T(y)$ if $x \geq y$. Deduce that T is continuous.